Controlling chaos in unidimensional maps using constant feedback

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We demonstrate that a constant feedback can control chaotic oscillations in one-dimensional discrete maps. This is a simple method of controlling chaos as the feedback signal does not require *a priori* knowledge of the dynamics of the system and it generates the desired behavior by simply varying the strength of the feedback. We illustrate this method with an application to the quadratic and exponential logistic maps.

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Much attention has been focused on controlling chaotic behavior exhibited by many nonlinear dynamical systems in recent times, and several algorithms [1–15] have been proposed for achieving control of chaos. In general, there are two ways to control chaos. In one way control is achieved by converting chaotic behavior into any of the desired periodic behaviors exhibited by the system. This kind of control algorithm may be classified as *suppressing chaos* algorithms and a few well-known algorithms of this kind are (i) the adaptive control algorithm [2,3], (ii) the effect of resonant parametric perturbation [4,5], (iii) the effect of second periodic force [6], (iv) weak feedback control [7], and (v) addition of noise [8].

The other way to control chaos is by stabilizing the unstable periodic orbits embedded in a chaotic attractor to any desired periodic orbit. Such prescriptions of control may be classified as *stabilizing chaos* algorithms. A typical example is the method developed by Ott, Grebogi, and Yorke, popularly known as the OGY algorithm [1,9], in which the stabilization of unstable periodic orbits associated with a chaotic attractor is achieved by applying small appropriate feedback to one of the accessible system parameters. More recently, other control algorithms of this kind such as proportional feedback [10], occasional proportional feedback [11,12], continuous feedback [13,14], and pulsed proportional feedback [15] algorithms have also been employed to control chaos by stabilizing unstable periodic orbits. All these control algorithms have been successfully applied to gain control over chaotic oscillations observed in both theoretical models and experimental systems. These studies have also been applied to other situations such as the synchronization of chaotic systems [16], the direction of trajectories to specified targets [17], and the transmission of information [18].

Most of the above mentioned control methods require *a priori* knowledge of the dynamics of the system, such as the location of stable fixed points or periodic attractors, and the choice of the feedback signal is based on this information. In many algorithms, the feedback signal is used to perturb some of the system parameters directly to control chaos in the system. In this Brief Report we present a simple method to control chaotic oscillations in unidimensional discrete maps. Our method can be classified as a suppressing chaos algorithm, as it converts a chaotic attractor into any desired periodic or fixed point attractor by simply varying the strength of a constant signal fed into the system at every iteration. This constant signal, which we refer to as the "feedback signal" neither requires any *a priori* knowledge of the dynamics of the system, such as the stable fixed points or unstable periodic orbits, nor changes any of the system parameters explicitly.

To describe our method, let us consider a typical one-dimensional discrete map of the form

$$x_{n+1}=f(x_n)\ , \tag{1}$$

where $f(x_n)$ is a nonlinear function possessing "single-humped" shape controlled by a parameter. Our simple control algorithm essentially consists of the application of a constant feedback to Eq. (1) at every iteration having the form

$$x_{n+1}=f(x_n)+k\ , \tag{2}$$

where k represents the strength of the feedback which can take both negative and positive values. In other words, the map is controlled by changing x_n in such a way that a constant value k is withdrawn (for $k<0$) from or injected (for $k>0$) into the value of x_n at every iteration. For a given map, this constant feedback (2) may be applied suitably to gain control over the chaotic dynamics exhibited by the system. However, the choice of the type of the feedback (i.e., negative or positive) to be used for controlling chaos depends on the response of the system to the applied feedback. Therefore this controlling algorithm is not equivalent to driving an unstable system to a quiescent state by increasing the dissipation as in the case of continuous systems, since for some maps control can be achieved only by applying negative feedback.

To illustrate our control algorithm we consider the quadratic logistic map [19–21]

$$x_{n+1}=rx_n(1-x_n) \tag{3}$$

and the exponential map [20,22,23]

$$x_{n+1}=x_n\exp[r(1-x_n)]\ , \tag{4}$$

where r is the parameter of the system. Equations (3) and (4) are widely used as population growth models in ecology [19–23] and they are considered as simple models for illustrating the occurrence of chaos in unidimensional maps [19].

In Fig. 1 we show the successive iterates (x_n vs n plot) of the quadratic logistic map (4) for $r=3.8$ subjected to a constant feedback (2) for different values of the feedback strength (k). When feedback is absent ($k=0$), system (3) exhibits chaotic oscillations as shown in Fig. 1(a). The corresponding Liapunov exponent in this case is calculated to be 0.43, which is positive as expected for chaotic oscillations. However, when the feedback control is switched on with a strength $k=-0.2$, the system exhibits period-2 ($P=2$) oscillations, which is shown in Fig. 1(b) and the chaotic behavior exhibited by the system shown in Fig. 1(a) is completely suppressed. Further decrease in the feedback strength to $k=-0.3$ results in a stable period-1 ($P=1$) or fixed point attractor, as shown in Fig. 1(c). the Liapunov exponents for the above values of $k=-0.2$ and -0.3 are found to be -0.80 and -0.21, respectively, and their negativity confirms the observed regular behavior in Figs. 1(b) and 1(c). From Fig. 1 it is evident that a constant feedback (with $k<0$) controls chaotic oscillations in the quadratic logistic map.

Figure 2 illustrates the control of chaotic dynamics for the exponential map (4) under the influence of a constant feedback (2) with strength $k>0$. Here the chaotic behavior exhibited by the system (4) at $r=2.8$ for $k=0$ [cf. Fig. 2(a)] is converted into period-2 and period-1 oscillations as the feedback strength is increased to $k=0.5$ [cf. Fig. 2(b)] and $k=0.95$ [cf. Fig. 2(c)], respectively. The corresponding Liapunov exponents of the exponential map (4) at $r=2.8$ are found to be 0.23, -0.84, and -0.06 for $k=0.0$, $k=0.5$, and $k=0.95$, respectively, confirming the dynamical behavior shown in Fig. 2. Thus it is clear that a constant feedback with strength $k>0$ is capable of controlling chaos in the exponential map.

We now describe the mechanism of controlling chaos involved in this method by using a simple geometrical interpretation. As an example, we consider the logistic map (3) exhibiting chaotic oscillations for $r=3.8$ in the absence of any feedback, i.e., $k=0$ [cf. Fig. 1(a)] which has been controlled by applying a constant feedback $k=-0.3$ [cf. Fig. 1(c)]. To illustrate this geometrically, we have obtained the x_{n+1} versus x_n plot of the logistic map for $r=3.8$ without feedback $k=0$ (broken-line curve) and with feedback $k=-0.3$ (solid-line curve) as shown in Fig. 3, where the bisecting line $x_{n+1}=x_n$ is depicted by the dotted line. In general, the evolution of x_n for every iteration is obtained by starting with some initial x_n value and moving vertically to the curve to obtain the x_{n+1} value. We then proceed horizontally until the $x_{n+1}=x_n$ line and use this value for the next iteration, from where a vertical move to the curve gives the next x_{n+1} value. Repeating this process for a number of iterations allows us to visualize the dynamics of the system. It can be easily seen (cf. Fig. 3) that discrete jumps (from the upper broken-line curve to the lower solid-line curve) occur in the value of x_{n+1} at every iteration as the constant feedback k is applied by the control algorithm.

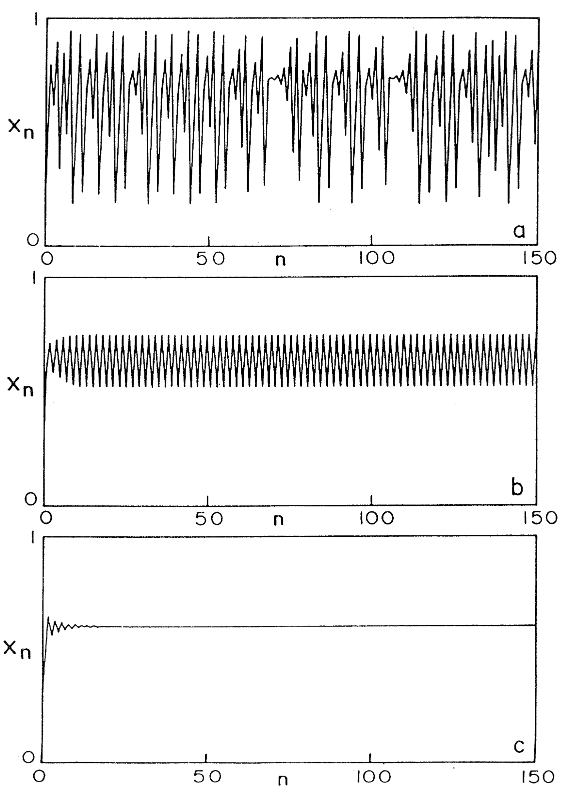

FIG. 1. Iterates of quadratic logistic map (3) subjected to constant feedback (2) for $r=3.8$ with (a) $k=0.0$, (b) $k=-0.2$, and (c) $k=-0.3$.

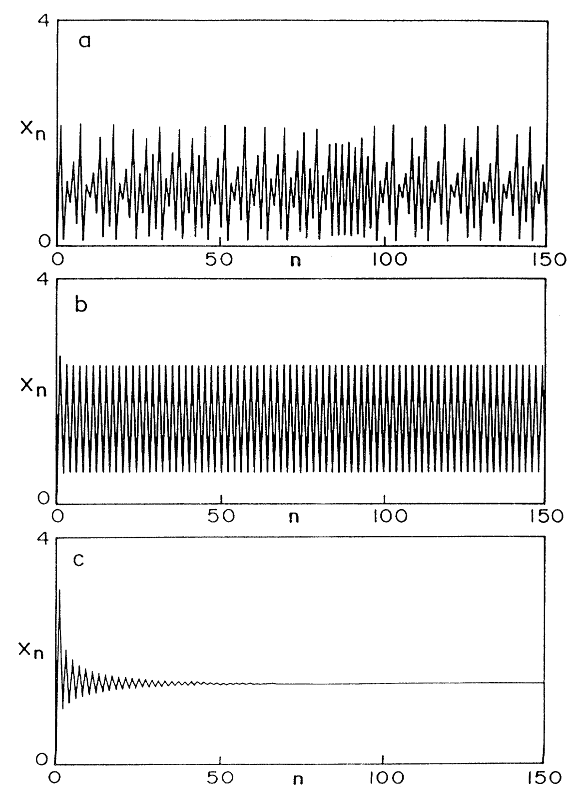

FIG. 2. Iterates of exponential map (4) subjected to constant feedback (2) for $r=2.8$ with (a) $k=0.0$, (b) $k=0.5$, and (c) $k=0.95$.

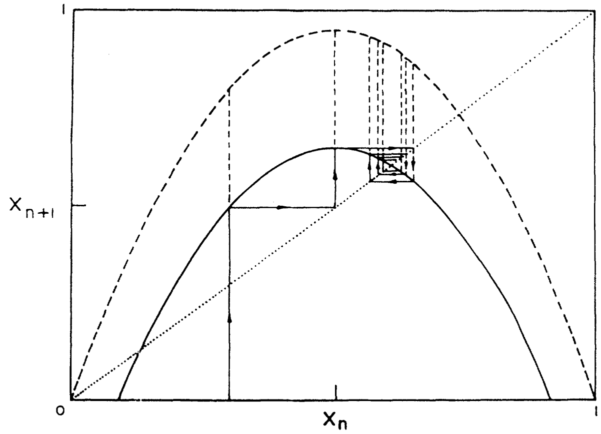

FIG. 3. Geometrical interpretation of the mechanism of control of chaos in logistic map (3) for $r = 3.8$ subjected to feedback with strength $k = -0.3$ (solid-line curve). Broken-line curve indicates the behavior of the system in the absence of feedback.

Thus if one follows the solid lines of Fig. 3, it is apparent that the effect of this control algorithm is to suppress the chaotic oscillations by changing the x_{n+1} value at every iteration resulting in a stable equilibrium behavior.

As mentioned earlier, to control a given system, the choice of feedback, i.e., negative or positive type, can be made by analyzing how the system responds to the applied feedback. For example, in the case of quadratic logistic map (3) control of chaos is obtained by using negative feedback, while the control algorithm uses a positive feedback for controlling the exponential map (4). In general, the choice of the type of the feedback can be easily made by analyzing the behavior of the given system when subjected to both negative and positive feedback signals for a few typical values of the parameters. The more rigorous way would be to investigate the dynamic behavior of a given system over a wide range of values of the relevant parameters when subjected to the negative and positive feedback. We have demonstrated this for the quadratic and exponential logistic maps [Eqs. (3) and (4)] and Figs. 4 and 5 show the dynamics in the $(r\text{-}k)$ parameter space, respectively, where the feedback strength k assumes both negative and positive values. In general, the figures show one region where the system possesses stable dynamics, such as stable orbits of periods $P = 1$, 2, 4 or >4 including chaos, and the other region is termed the "escape" region, where $x_n \leq 0$, at any n, eventually leading the system to escape to $-\infty$. For the present purpose, the stable regions instead of the escape boundary are considered as the useful range of parameters displaying a wide variety of ordered and chaotic dynamics. We have used the initial value x_0 as 0.3, for all our calculations.

Figure 4 outlines the behavior of the logistic map (3) over a range of values in r-k parameter space subjected to the constant feedback (2) and demarcates regions in which the system exhibits stable orbits of periods $P = 1$, 2, 4, and >4 including chaos. This stable region is bounded by the escape regions for both the negative and positive values of k. It is clear from Fig. 4 that just below the region with $P > 4$, where the system exhibits oscillations with periods greater than 4 including chaos, we have regions with periods $P = 4$ or 2 or 1, with decreasing values of k. This property of the logistic map (3) with the applied feedback has been used to suppress chaotic dynamics by applying appropriate negative feedback. Thus it is clear that the control of chaos in system (3) is achieved through period-doubling reversals under the influence of the constant negative feedback.

In a similar way, Fig. 5 outlines the behavior of the exponential map (4) over a range of r-k parameter space with constant feedback. In contrast to Fig. 4, it is evident from Fig. 5 that the exponential map (4) possesses an escape region only for negative values of k. Further, this system (4) exhibits an increasing level of complicated behavior as the parameter r increases and this complicated behavior can be controlled by using both positive and

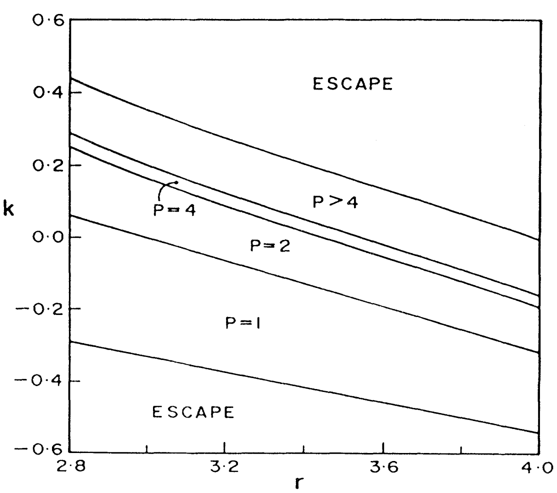

FIG. 4. Regions in r-k parameter space, characterized numerically, for oscillations with periods $P = 1,2,4, >4$ for the quadratic logistic map (3) with constant feedback.

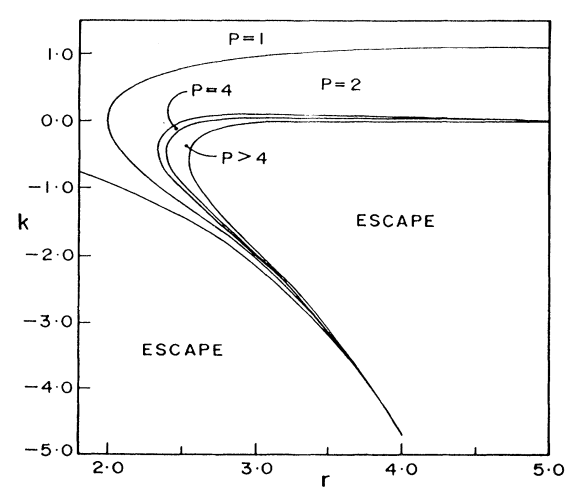

FIG. 5. Regions in r-k parameter space, characterized numerically, for oscillations with periods $P = 1,2,4, >4$ for the exponential map (4) with constant feedback.

negative feedback signals. For example, we have already shown in Fig. 2 that the system (4) exhibits chaotic oscillations for $r=2.8$ in the absence of feedback ($k=0$) which has been controlled to period-2 and period-1 oscillations by applying the feedback (2) with positive values of $k=0.5$ and $k=0.95$, respectively. Similar control of chaos can also be achieved for the system (4) for the same value of $r=2.8$ by applying constant feedback with negative values of $k=-1.595$ and $k=-1.695$, which results in period-2 and period-1 oscillations, respectively. However, Fig. 5 clearly indicates that the exponential map (4) with negative feedback strength ($k<0$) exhibits stable oscillations including chaos within a narrow wedge-shaped band of k values. As the band becomes narrow for increasing r, the range of k values used for controlling chaos correspondingly becomes small. On the other hand, the behavior of the exponential map (4) with positive feedback strength ($k>0$) exhibits the $P=2$ and 1 regions for a wide range of k, even for high r values. Thus in the case of the exponential map control of chaos can be easily achieved with the help of a constant positive feedback.

It may be mentioned here that we have also applied this control method to other one-dimensional discrete maps [19,20] such as

$$x_{n+1}=rx_n/[1+(ax_n)^b] \tag{5}$$

and

$$x_{n+1}=rx_n/(1+ax_n)^b , \tag{6}$$

where r, a, and b are parameters. We find that the control of chaos can easily be achieved in these maps with the help of constant positive feedback. More recently, a similar kind of control algorithm has also been successfully applied to control the chaotic oscillations exhibited by certain continuous systems described by coupled ordinary differential equations, such as the forced Duffing-Holmes oscillator [24] and certain nonlinear electronic circuits [25], both experimentally and numerically.

In conclusion, we have shown that it is possible to control one-dimensional discrete maps in a range of parameters that makes the system exhibit deterministic chaos by simply applying a constant feedback with negative or positive amplitude at every iteration. The suggested method does not require any *a priori* knowledge of the dynamics of the system for its feedback signal and also it does not alter any of the system parameters explicitly. The algorithm is applied to the quadratic and exponential logistic maps for various values of parameters. Finally, we may mention that in this simple method the effect of applying a constant feedback for a given system either can produce a resultant system that can exhibit different dynamical properties from the original one or can be scaled out leaving the system invariant, as in the case of the quadratic logistic map [22].

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